DIFFERENTIAL MANIFOLDS HW 1

KELLER VANDEBOGERT

1. Exercise 1.7

Define $p_k := x^{(k)}$. Then, for $0 \leq k < n-1$, $\frac{dp_k}{dt} = p_{k+1}$ and $\frac{dp_{n-1}}{dt} = F(t,\ p_0,\ \dots\ ,\ p_{n-1})$.

Define $(t,\ p_0,\ \dots\ ,\ p_{n-1}) \mapsto \big(1,\ p_1,\ \dots\ , F(t,\ p_0,\ \dots\ ,\ p_{n-1})\big)$ as the mapping V. Then it is clear by our definitions that if $x := (t,\ p_0,\ \dots\ ,\ p_{n-1})$, x satisfies

$$\frac{dx}{dt} = V(x)$$

2. Exercise 1.12

(a). Using (1.26) from the notes, we have:

$$\mathrm{Ad}_F U(x) = DF(x)(U(x))$$

So that by definition,

$$\begin{aligned} DF(x)(U(x)) &= DF(x)(x^2) \\ &= \lim_{t\to 0} \frac{\frac{1}{x+tx^2} - \frac{1}{x}}{t} \\ &= \lim_{t\to 0} \frac{-tx^2}{(x+tx^2)tx} = \frac{-x^2}{x^2} = -1 \end{aligned} \tag{2.1}$$

Implying that $\mathrm{Ad}_F U(x) = -1$.

Date: September 3, 2017.

(b). Now, to compute the Lie bracket, employ (1.27), and also use the fact that for very small ϵ, $(1+\epsilon)^n \approx 1 + n\epsilon$. We have the following:

$$\begin{aligned} DV(x)(W(x)) &= \lim_{t\to 0} \frac{(x+tx^q)^p - x^p}{t} \\ &= \lim_{t\to 0} \frac{x^p(1+ptx^{q-1}) - x^p}{t} \\ &= \lim_{t\to 0} \frac{ptx^{p+q-1}}{t} = px^{p+q-1} \end{aligned} \tag{2.2}$$

And similarly,

$$\begin{aligned} DW(x)(V(x)) &= \lim_{t\to 0} \frac{(x+tx^p)^q - x^q}{t} \\ &= \lim_{t\to 0} \frac{x^q(1+qtx^{p-1}) - x^q}{t} \\ &= \lim_{t\to 0} \frac{qtx^{p+q-1}}{t} = qx^{p+q-1} \end{aligned} \tag{2.3}$$

And upon taking the difference, we see that

$$[V,W](x) = (p-q)x^{p+q-1}$$

3. Exercise 1.13

(a). Suppose $F : X \to Y$ is a diffeomorphism. Then, $y = F(x)$, and consider the vector field $y \mapsto \frac{\partial y}{\partial x^i}$.

Then, $\frac{dy}{dt} = DF(x)(\frac{dx}{dt})$, and $\frac{\partial y}{\partial x^i} = DF(x)(e_i)$. Now, to calculate the flow, we set these equal. By linearity, we have:

$$DF(x)(\frac{dx}{dt} - e_i) = 0$$

Since x is arbitrary and F is a diffeomorphism, we conclude that $\frac{dx}{dt} - e_i = 0$ for all x. But this then implies we have the following flow for x:

$$x(t) = te_i + C$$

Where C is a constant vector. Then, since $F(x) = y$, we can calculate the flow at y:

$$e^{t\frac{\partial}{\partial x^i}}(a) = F(te_i + F^{-1}(a))$$

(b). Using the definition of the Lie Bracket, we have:

$$\begin{aligned}
\left[\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right] &= \frac{d}{ds}\frac{d}{dt} e^{s\frac{\partial}{\partial x^i}} \circ e^{t\frac{\partial}{\partial x^j}} \circ e^{-s\frac{\partial}{\partial x^i}}(x) \\
&= \frac{d}{ds}\frac{d}{dt} e^{s\frac{\partial}{\partial x^i}} \circ e^{t\frac{\partial}{\partial x^j}} F(-se_i + F^{-1}(x)) \\
&= \frac{d}{ds}\frac{d}{dt} e^{s\frac{\partial}{\partial x^i}} F(te_j - se_i + F^{-1}(x)) \\
&= \frac{d}{ds}\frac{d}{dt} F(te_j + F^{-1}(x)) \\
&= 0
\end{aligned} \tag{3.1}$$

Where the final equality notes that the above no longer depends on s, so its derivative with respect to s is 0. Hence we see that the above vector fields commute with respect to the Lie Bracket (this makes sense formally, since we expect that $\frac{\partial}{\partial x^i}\frac{\partial}{\partial x^j} = \frac{\partial}{\partial x^j}\frac{\partial}{\partial x^i}$).

4. Exercise 1.17

By definition of Lie derivative and pullback operation, we have:

(4.1)

$$\begin{aligned}
L_V\omega(v_1, \ \dots \ , \ v_p) &= \frac{d}{dt} e^{tV*}\omega(v_1, \ \dots \ , \ v_p)\Big|_{t=0} \\
&= \frac{d}{dt}\omega_{e^{tV}(x)}(De^{tV}(x)(v_1), \ \dots \ , \ De^{tV}(x)(v_p))\Big|_{t=0} \\
&= \frac{d}{dt}\omega_{e^{tV}(x)}(D(e^{tV}(x))(v_1), \ \dots \ , \ D(e^{tV}(x))(v_p))\Big|_{t=0}
\end{aligned}$$

Since the above is multilinear, the derivative follows the Leibniz rule:

$$
\begin{aligned}
&\frac{d}{dt}\omega_{e^{tV}(x)}(D(e^{tV}(x))(v_1),\ \dots\ ,\ D(e^{tV}(x))(v_p))\Big|_{t=0} = \\
&\frac{\partial \omega}{\partial x}(V(e^{tV}(x))(D(e^{tV}(x))(v_1),\ \dots\ ,\ D(e^{tV}(x))(v_p)) \\
&\quad + \omega_{e^{tV}(x)}(D^2(e^{tV}(x))(v_1)(dx/dt),\ \dots\ ,\ D(e^{tV}(x))(v_p))\Big|_{t=0} \\
&\quad + \omega_{e^{tV}(x)}(D(e^{tV}(x))(v_1),\ D^2(e^{tV}(x))(v_2)(dx/dt),\ \dots\ ,\ D(e^{tV}(x))(v_p))\Big|_{t=0} \\
&\quad + \dots \\
&\quad + \omega_{e^{tV}(x)}(D(e^{tV}(x))(v_1),\ \dots\ ,\ D^2(e^{tV}(x))(v_p)(dx/dt))\Big|_{t=0}
\end{aligned} \tag{4.2}
$$

Now set $t = 0$ in the above, and we note that $e^{tV}(x) = \gamma_x(t)$, where $\gamma_x(0) = x$. For each v_i this then implies that $D^2(e^{tV}(x))(dx/dt)(v_i)|_{t=0} = D(de^{tV}dt(x))(v_i)|_{t=0} = DV(x)(v_i)$. Also note that $Dx(v_i) = \mathbb{I}v_i = v_i$. With this, we can simplify the above massively once $t = 0$.

$$
\begin{aligned}
L_V\omega(v_1,\ \dots\ ,\ v_p) =& \frac{\partial \omega}{\partial x}(V(x))(v_1,\ \dots\ ,\ v_p) \\
&+ \omega(DV(x)(v_1),\ \dots\ ,\ v_p) \\
&+ \dots \\
&+ \omega(v_1,\ \dots\ ,\ DV(x)(v_p))
\end{aligned} \tag{4.3}
$$

And we are done.

5. Exercise 1.19

(a). We employ E. Cartan's formula:

$$
\begin{aligned}
\mathrm{d}L_V\omega &= \mathrm{d}(\mathrm{d}i_V\omega + i_V\mathrm{d}\omega) \\
&= \mathrm{d}^2 i_V\omega + \mathrm{d}i_V\mathrm{d}\omega \\
&= \mathrm{d}i_V\mathrm{d}\omega + i_V\mathrm{d}\mathrm{d}\omega \\
&= L_V\mathrm{d}\omega
\end{aligned} \tag{5.1}
$$

Where we've used that $\mathrm{d}^2 = 0$ twice in the above.

(b). Note that for vector fields U, V, we have that $L_U V = [U, V]$. Using this and Theorem 8.55 of Lee, we have:

$$
\begin{aligned}
L_U(i_V\omega) &= i_{L_U V}\omega + i_V L_U\omega \\
&= i_{[U,V]}\omega + i_V L_U\omega
\end{aligned} \tag{5.2}
$$

Now, merely subtracting we find that $i_{[U,V]}\omega = L_U i_V\omega - i_V L_U\omega$, as desired.

(c). Again using Cartan's magic formula and the result of parts (a) and (b):

$$
\begin{aligned}
L_{[U,V]}\omega &= \mathrm{d}i_{[U,V]}\omega + i_{[U,V]}\mathrm{d}\omega \\
&= \mathrm{d}(L_U i_V\omega - i_V L_U\omega) + L_U i_V\mathrm{d}\omega - i_V L_U\mathrm{d}\omega \\
&= L_U\mathrm{d}i_V\omega + L_U i_V\mathrm{d}\omega - (\mathrm{d}i_V L_U\omega + i_V\mathrm{d}L_U\omega) \\
&= L_U(\mathrm{d}i_V\omega + i_V\mathrm{d}\omega) - (\mathrm{d}i_V + i_V\mathrm{d})(L_U\omega) \\
&= L_U L_V\omega - L_V L_U\omega
\end{aligned} \tag{5.3}
$$